Spherically Symmetric, Self-Similar Spacetimes

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Self-similar spacetimes are of importance to cosmology and to gravitational collapse problems. We show that self-similarity or the existence of a homothetic Killing vector field for spherically symmetric spacetimes implies the separability of the spacetime metric in terms of the co-moving coordinates and that the metric is, uniquely, the one recently reported in [5]. The spacetime, in general, has non-vanishing energy flux and shear. The spacetime admits matter with *any* equation of state.

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A self-similar spacetime is characterized by the existence of a Homothetic Killing vector field [1]. For such a spacetime all points along the integral curves of the Homothetic Killing vector field are similar. Any spherically symmetric spacetime is self-similar if it admits a radial area coordinate r and an orthogonal time coordinate τ such that for the metric components $g_{\tau\tau}$ and g_{rr} the following holds

$$g_{\tau\tau}\,(\,\kappa\tau,\,\kappa r\,)\;=g_{\tau\tau}\,(\,\tau,\,r\,) \tag{1}$$

$$g_{rr}\,(\,\kappa\tau,\,\kappa r\,)\;=g_{rr}\,(\,\tau,\,r\,) \tag{2}$$

for all constants $\kappa > 0$. Consequently, for a self-similar spacetime, the Einstein field equations reduce to ordinary differential equations.

The main application of self-similar spacetimes has been in the cosmological context [1–3]. Such spacetimes can also describe the gravitational collapse if suitable matching can be done with an appropriate exterior spacetime [4,5]. While self-similarity is a strong restriction of a geometric nature on the spacetime, it has been successfully exploited in various physical scenarios. (See [6] for a recent survey on the importance of self-similarity in General Relativity.)

A Conformal Killing Vector (CKV) $\mathbf{X}$ satisfies

$$\mathcal{L}_{\mathbf{X}}\,g_{ij}\;=\;2\,\Phi(x^k)\,g_{ij} \tag{3}$$

where $\Phi(x^k)$ is the conformal factor and g_{ij} is the spacetime metric tensor. In the case of Homothetic Killing vectors, Φ must be a constant. Such vectors scale distances by the same constant factor and also preserve the null geodesic affine parameters.

In what follows, we will demand that a spherically symmetric spacetime admits a Homothetic Killing vector of the form

$$X^a \;=\; (0, f(r,t), 0, 0) \tag{4}$$

Typically, the homothetic Killing vector in suitable coordinates is taken to be [2]:

$$\bar{X}^a \;=\; (t, r, 0, 0) \tag{5}$$

However, any vector of the form (4) can be transformed into (5) via a coordinate transformation of the form

$$R = l(t) \exp\left(\int f^{-1} dr\right) \qquad T = k(t) \exp\left(\int f^{-1} dr\right) \tag{6}$$

and so we will not loose any generality[1] by concentrating on (4).

If we require that the general spherically symmetric line element

$$ds^2 \;=\; -e^{2\nu(t,r)} dt^2 \,+\, e^{2\lambda(t,r)} dr^2 \,+\, Y^2(t,r)(d\theta^2 + \sin^2\theta d\phi^2) \tag{7}$$

admits (4), the expression (3) reduces to the system of four equations:

$$f(r,t)\,\frac{\partial\,\nu}{\partial r} = \Phi \tag{8}$$

$$\frac{\partial\, f(r,t)}{\partial t} = 0 \tag{9}$$

$$\left(\frac{1}{Y}\frac{\partial\, Y}{\partial r} - \frac{\partial\,\nu}{\partial r}\right) f(r,t) = 0 \tag{10}$$

$$\left(\frac{\partial\,\lambda}{\partial r} - \frac{\partial\,\nu}{\partial r}\right) f(r,t) \,+\, \frac{\partial\, f(r,t)}{\partial r} = 0 \tag{11}$$

where Φ is a constant. (See [7] for a comprehensive description of the general conformal geometry of (7).)

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[1]Note: The general metric admitting (5) has metric functions which are functions of t/r. If we invoke (6), the resulting metric will not be diagonal. The imposition of diagonality of the metric will require a relationship between $l(t)$ and $k(t)$. Such a relation can always be imposed.

Solving the above system of equations, we obtain

$$f = F(r) \tag{12}$$

$$Y = \tilde{g}(t)\,\exp\left(\int \frac{\Phi}{F(r)} dr\right) \tag{13}$$

$$\lambda = \int \frac{\Phi}{F(r)} dr \;-\; \log F(r) \;+\; \tilde{h}(t) \tag{14}$$

$$\nu = \int \frac{\Phi}{F(r)} dr + \tilde{k}(t) \tag{15}$$

so that the spacetime metric becomes *separable* and is given by

$$ds^2 \;=\; k^2(t)\exp\left(2\int \frac{\Phi}{F(r)} dr\right)\left[-dt^2 \;+\; \frac{h^2(t)}{F^2(r)}\,dr^2 \;+\; g^2(t)\left(d\theta^2 \;+\; \sin^2\theta\, d\phi^2\right)\right] \tag{16}$$

One would now substitute this metric into the Einstein Field Equations in order to determine the explicit forms of the metric functions for the choice of the energy-momentum tensor for the matter in the spacetime. However, this metric has already been obtained in [5] as:

$$ds^2 \;=\; -\,y^2(r)\,A^2(t)\,dt^2 \;+\; \gamma^2\,B^2(t)\,\left(\frac{dy}{dr}\right)^2\,dr^2 \;+\; y^2(r)\,R^2(t)\left[\,d\theta^2 \;+\; \sin^2\theta\,d\phi^2\,\right] \tag{17}$$

where $y(r)$ is an arbitrary function of r.

(Using the differential equations solver in `PROGRAM LIE` [8], it is easy to verify that the Homothetic Killing vector

$$X = \frac{y}{\Phi\, y'}\frac{\partial}{\partial r} \tag{18}$$

is admitted by (17). See also [9].) The two metrics (16) and (17) are clearly identical.

We note that the requirement of the self-similarity of the spherically symmetric spacetime *uniquely* fixes the metric to (17). All metrics which admit a Homothetic Killing vector will be contained in this metric (or can be transformed to this form). The spacetime of metric (17) admits any equation of state for the matter in the spacetime. This spacetime was obtained by first imposing the separability of the metric functions and then imposing the field equations. *It now follows that the assumption of separability of the metric functions for a spherically symmetric spacetime and the satisfaction of the field equations provide us with a self-similar spacetime and vice-versa.*

The spacetime (17) is radiating and shearing. All spacetimes admitting a Homothetic Killing vector, that have vanishing or non-vanishing shear and/or heat flux, *must* be contained with this spacetime.

The crucial point here is that, in those cases, more field equations need to be solved which further constrains the metric functions. However, the equivalence of the spacetimes only holds if (6) is non-singular. For example, in the case of the Robertson-Walker spacetime, the transformation is singular as the function $F(r) = 0$.

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[2] See also Joshi, P. S. (1993) *Global Aspects in Gravitation and Cosmology* (Clarendon Press, Oxford) and references therein.

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